

# Spy-Game on graphs: complexity and simple topologies \*

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## Abstract

We define and study the following two-player game on a graph  $G$ . Let  $k \in \mathbb{N}^*$ . A set of  $k$  *guards* is occupying some vertices of  $G$  while one *spy* is standing at some node. At each turn, first the spy may move along at most  $s$  edges, where  $s \in \mathbb{N}^*$  is his *speed*. Then, each guard may move along one edge. The spy and the guards may occupy the same vertices. The spy has to escape the surveillance of the guards, i.e., must reach a vertex at distance more than  $d \in \mathbb{N}$  (a predefined distance) from every guard. Can the spy win against  $k$  guards? Similarly, what is the minimum distance  $d$  such that  $k$  guards may ensure that at least one of them remains at distance at most  $d$  from the spy? This game generalizes two well-studied games: Cops and robber games (when  $s = 1$ ) and Eternal Dominating Set (when  $s$  is unbounded).

We consider the computational complexity of the problem, showing that it is NP-hard (for every speed  $s$  and distance  $d$ ) and that some variant of it is PSPACE-hard in DAGs. Then, we establish tight tradeoffs between the number of guards, the speed  $s$  of the spy and the required distance  $d$  when  $G$  is a path or a cycle.

**Keywords:** Cops and Robber games, graphs, PSPACE-hard

## 1 Introduction

Two-player turn-by-turn games on graphs have been well studied with two of the well-known games being the *Cops and robber game* [18, 7] and the *eternal domination game* [13]. There exist many open problems in these two games and in this paper, we define and study a generalization of these games. We believe that our game can lead to results of some of these open problems, much like other variants (see [6, 12, 2, 8, 10]) which have been studied for the same purpose. For example, a strategy for the guards in our game may be adapted to a strategy for the guards in the eternal domination game. As well, the motivation behind the study of the complexity of our game is that, as far as we know, it is not known whether the eternal domination game is PSPACE-hard (it is known to be NP-hard, but it is not known whether it is in NP).

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We consider the following two-player game on a graph  $G$ , called *Spy-game*. Let  $k, d, s \in \mathbb{N}$  be three integers such that  $k > 0$  and  $s > 0$ . One player uses a set of  $k$  *guards* occupying some vertices of  $G$  while the other player plays with one *spy* initially standing at some node. This is a full information game, thus any player has full knowledge of the positions and previous moves of the other player. Note that several guards and even the spy could occupy the same vertex.

Initially, the spy is placed at some vertex of  $G$ . Then, the  $k$  guards are placed at some vertices of  $G$ . Then, the game proceeds turn-by-turn. At each turn, first the spy may move along at most  $s$  edges ( $s$  is the *speed* of the spy). Then, each guard may move along one edge. The spy wins if, after a finite number of turns (after the guards' move), it reaches a vertex at distance greater than  $d$  from every guard. The guards win otherwise, in which case we say that the guards *control* the spy at distance  $d$ , i.e. there is always at least one guard at distance at most  $d$  from the spy.

Given a graph  $G$  and two integers  $d, s \in \mathbb{N}$ ,  $s > 0$ , let the *guard-number*, denoted by  $gn_{s,d}(G)$ , be the minimum number of guards required to control a spy with speed  $s$  at distance  $d$ , against all spy's strategies.

## 1.1 Preliminary remarks

We could define the game by placing the guards first. In that case, since the spy could choose its initial vertex at distance greater than  $d$  from any guard, we need to slightly modify the rules of the game. If the guards are placed first, they win if, after a finite number of turns, they ensure that the spy always remains at distance at most  $d$  from at least one guard. Equivalently, the spy wins if it can reach infinitely often a vertex at distance greater than  $d$  from every guard. We show that both versions of the game are closely related. In what follows, we consider the spy-game against a spy with speed  $s$  that must be controlled at distance  $d$  for some fixed integers  $s > 0$  and  $d$ .

**Claim 1.** *If the spy wins against  $k$  guards in the game when it starts first, then it wins in the game when it is placed after the  $k$  guards.*

*Proof of the claim.* Assume that the spy has a winning strategy  $\mathcal{S}$  when it is placed first. In particular, there is a vertex  $v_0 \in V(G)$  such that, starting from  $v_0$  and whatever be the strategy of the guards, the spy can reach a vertex at distance  $> d$  from every guard. If the spy is placed after the guards, its strategy first consists in reaching  $v_0$ , and then in applying the strategy  $\mathcal{S}$  until it is at distance  $> d$  from every guard. The spy repeats this process infinitely often.  $\diamond$

The converse is not necessarily true, however we can prove a slightly weaker result which is actually tight. For this purpose, let us recall the definition of the well known *Cops and robber* game [18, 7]. In this game, first  $k$  cops occupy some vertices of the graph. Then, one robber occupies a vertex. Turn-by-turn, each player may move its token (the cops first and then the robber) along an edge. The cops win if one of them reaches the same vertex as the robber after a finite number of turns. The robber wins otherwise. The *cop-number*  $cn(G)$  of a graph  $G$  is the minimum number of cops required to win in  $G$  [1].

**Claim 2.** *If  $k$  guards win in the game when the spy is placed first in a graph  $G$ , then  $k + cn(G) - 1$  guards win the game when they are placed first.*

*Proof of the claim.* Assume that  $k$  guards have a winning strategy when the spy is placed first. Such a strategy  $\mathcal{S}$  is defined as follows. For any walk  $W = (v_0, v_1, \dots, v_\ell)$  of the spy<sup>1</sup>, each guard  $g_i$  ( $1 \leq i \leq k$ ) is assigned a vertex  $pos_i(W)$ , such that, for any vertex  $w \in V(G)$  at distance at most  $s$  from  $v_\ell$  and for any  $i \leq k$ ,  $pos_i(W \cdot w) \in N[pos_i(W)]$  where  $N[x]$  denote the

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<sup>1</sup>Here, a walk is a sequence of vertices (possibly with repetitions) such that two consecutive vertices in the sequence are at distance at most  $s$ .

set of vertices at distance at most one from  $x \in V$ . Moreover, for any walk  $W = (v_0, \dots, v_\ell)$  of the spy, there exists  $i \leq k$  such that the distance between  $v_\ell$  and  $\text{pos}_i(W)$  is at most  $d$ .

Now, let us assume that  $k + \text{cn}(G) - 1$  guards are placed first. We show that after a finite number of turns, when the spy has followed any walk  $W$ , the vertices  $\text{pos}_i(W)$  are occupied for all  $1 \leq i \leq k$  and then the guards occupying these vertices can follow  $\mathcal{S}$  and so win.

Let  $0 \leq j < k$  and assume that the spy has followed the walk  $W = (v_0, \dots, v_\ell)$  (in particular, the spy occupies  $v_\ell$ ) and that the vertices  $\text{pos}_i(W)$  are occupied for all  $1 \leq i \leq j$  ( $j = 0$  means no such vertex is occupied). The guards occupying the vertices  $\text{pos}_1(W), \dots, \text{pos}_j(W)$  follow their strategy  $\mathcal{S}$ . There remains  $k + \text{cn}(G) - 1 - j \geq \text{cn}(G)$  “free” guards. A team of  $\text{cn}(G)$  of free guards will target the position  $\text{pos}_{j+1}(W)$  (which acts as a robber moving at speed one in  $G$ ). Therefore, after a finite number of steps, one free guard reaches  $\text{pos}_{j+1}(W')$  (where  $W'$  is the walk that the spy has followed until this step). Continuing this way, after a finite number of steps, after that the spy has followed some walk  $W^*$ , the vertices  $\text{pos}_i(W^*)$  are occupied for all  $1 \leq i \leq k$ . These  $k$  guards can go on executing  $\mathcal{S}$  and win, which concludes the proof.  $\diamond$

The bound of the previous claim is tight. Indeed, for any graph  $G$ ,  $gn_{1,0}(G) = 1$  since one guard can be placed at the initial position of the spy and then follow it. On the other hand, if the guards are placed first, the game (for  $s = 1$  and  $d = 0$ ) is equivalent to the classical Cops and robber game and, therefore,  $\text{cn}(G)$  guards are required.

## 1.2 Related work

**Further relationship with Cops and robber games.** The Cops and robber game has been generalized in many ways [6, 12, 2, 8, 10]. In [6], Bonato *et al.* proposed a variant with *radius of capture*. That is, the cops win if one of them reaches a vertex at distance at most  $d$  (a fixed integer) from the robber. The version of our game when the guards are placed first and for  $s = 1$  is equivalent to Cops and robber with radius of capture. Indeed, when the spy is not faster than the guards, capturing the spy (at any distance  $d$ ) is equivalent to controlling it at such a distance: once a guard is at distance at most  $d$  from the spy, it can always maintain this distance (by following a shortest path toward the spy).

This equivalence is not true anymore as soon as  $s > 1$ . Indeed, one cop is always sufficient to capture one robber in any tree, whatever be the speed of the robber or the radius of capture. On the other hand, we prove below that  $\Theta(n)$  cops are necessary to control a spy with speed at least 2 at some distance  $d$  in any  $n$ -node path. This is mainly due to the fact that, in the spy-game, the spy may cross (or even occupy) a vertex occupied by a guard. Therefore, in what follows, we only consider the case  $s \geq 2$ .

Note that the Cops and robber games when the robber is faster than the cops is far from being well understood. For instance, the exact number of cops with speed one required to capture a robber with speed two is unknown in 2-dimensional grids [11, 5]. One of our hopes when introducing the Spy-game is that it will lead us to a new approach to tackle this problem.

**Generalization of Eternal Domination.** A  $d$ -dominating set of a graph  $G$  is a set  $D \subseteq V(G)$  of vertices such that any vertex  $v \in V(G)$  is at distance at most  $d$  from a vertex in  $D$ . Let  $\gamma_d(G)$  be the minimum size of a  $d$ -dominating set in  $G$ . Clearly,  $gn_{s,d}(G) \leq \gamma_d(G)$  for any  $s, d \in \mathbb{N}$ . However these two parameters may differ arbitrarily as shown by the following example. Let  $G$  be the graph obtained from a cycle  $C$  on  $n$ -vertices by adding one node  $x$  and, for any  $v \in C$ , adding a path of length  $d + 1$  between  $v$  and  $x$ . It is easy to check that  $\gamma_d(G) = \Omega(n/d)$  while  $gn_{s,d}(G) = 2$  (the two guards moving on  $x$  and its neighbors).

In the *eternal domination* game [13, 14, 16, 17], a set of  $k$  *defenders* occupy some vertices of a graph  $G$ . At each turn, an *attacker* chooses a vertex  $v \in V$  and the defenders may move to adjacent vertices in such a way that at least one defender is at distance at most  $d$  (a fixed predefined value) from  $v$ . Several variants of this game exist depending on whether exactly one

or more defenders may move at each turn [14, 16, 17]. It is easy to see that the spy-game, when the spy has unbounded speed (equivalently, speed at least the diameter of the graph) is equivalent to the Eternal Domination game when all defenders may move at each turn.

Eternal Domination and Spy game are also related to Patrolling games where a team of patrollers must move in a graph such that every vertex must never be unoccupied during more than  $d$  consecutive steps where  $d$  is a fixed parameter [4, 19]. In particular, since at each step, no vertex is at distance more than  $d$  from some patroller, the minimum size of a team for the Patrolling game provides an upper bound on the minimum number of guards required for controlling the spy at distance  $d$ , whatever be its speed.

### 1.3 Our contributions

In this paper, we initiate the study of the spy-game for  $s \geq 2$ . In Section 2, we study the computational complexity of the problem of deciding the guard-number of a graph. We prove that computing  $gn_{s,d}(G)$  is NP-hard for any  $s \geq 2$  and  $d \geq 0$ , in the class of graph  $G$  with diameter at most  $O(d)$ . Then, we show the problem is PSPACE-complete in the case of DAGs (where guards and spy have to follow the orientation of arcs, but distances are in the underlying graph). Then, we consider particular graph classes. In Section 3, we precisely characterize the cases of paths and cycles. Precisely, for any  $d \geq 0$ ,  $s \geq 2$ , we prove that, for any path  $P_n$  on  $n$  vertices:

$$gn_{s,d}(P_n) = \left\lceil \frac{n}{2d + 2 + \left\lfloor \frac{2d}{s-1} \right\rfloor} \right\rceil,$$

and, for any cycle  $C_n$  with  $n$  vertices:

- $gn_{s,d}(C_n) = \left\lceil \frac{n}{2d+3} \right\rceil$  if  $0 \leq 2d < s-1$ ;
- If  $2d \geq s-1$ , let  $2d = q(s-1) + r$  ( $0 \leq r < s-1$ ) and  $2d = q's + r'$  ( $0 \leq r' < s$ ). Let  $(q^*, r^*) = (q, r)$  if  $s$  odd and  $(q^*, r^*) = (q', r')$  otherwise. Then,

$$\left\lceil \frac{n + 2q}{2(d + q) + 3} \right\rceil \leq gn_{s,d}(C_n) \leq \left\lceil \frac{n + 2q^*}{2(d + q^*) - r^*} \right\rceil.$$

To conclude the paper, in Section 4, we also define another variant of the Spy-game and we initiate its study in paths.

**Notations.** We consider connected simple graphs. Given a graph  $G = (V, E)$  and  $v \in V$ , let  $N(v) = \{w \mid vw \in E\}$  denote the set of neighbors of  $v$  and let  $N[v] = N(v) \cup \{v\}$ .

## 2 Complexity

### 2.1 NP-hardness

In this section, we prove that the Spy-Game with speed  $s$  and distance  $d$  is NP-hard for any  $s \geq 2$  and  $d \geq 0$ . Precisely, we prove the following theorem.

**Theorem 3.** *Let  $s \geq 2$  and  $d \geq 0$  be two fixed integers. The problem that takes an  $n$ -node graph  $G$  and an integer  $k \in \mathbb{N}$  as inputs and aims at deciding whether  $gn_{s,d}(G) \leq k$  is NP-hard,  $W[2]$ -hard (see [9]) when parameterized by the number of guards, and  $\alpha \ln n$ -inapproximable in polynomial time for some constant  $0 < \alpha < 1$ , unless  $P=NP$ .*

The proof follows the five Lemmas below. The reduction is from the SET COVER Problem and is divided in three cases:  $d+1 < s < 2d+2$  (Lemma 5),  $s \geq 2d+2$  (Lemma 6), and  $s \leq d+1$  (Lemmas 8 and 9 depending on whether  $d \pmod{s-1}$  is greater than  $s/2$ ). The proofs of all

the cases are similar and vary slightly depending on the parameters. We present the proofs of all the cases separately for better comprehensibility.

An instance of the SET COVER Problem is a family  $\mathcal{S} = \{S_1, \dots, S_m\}$  of sets and an integer  $c$ , and the objective is to decide if there exists a subfamily  $\mathcal{C} = \{S_{i_1}, \dots, S_{i_c}\} \subseteq \mathcal{S}$  such that  $|\mathcal{C}| \leq c$  and  $S_{i_1} \cup \dots \cup S_{i_c} = U$ , where  $U = S_1 \cup \dots \cup S_m$  (we say that  $\mathcal{C}$  is a set cover of  $U$ ). Given an instance  $(\mathcal{S}, c)$  of Set Cover, we construct a graph  $G = G_{s,d}(\mathcal{S}, c)$  and an integer  $K = K_{s,d}(\mathcal{S}, c)$  such that there exists a cover  $\mathcal{C} \subseteq \mathcal{S}$  of  $U$  with size at most  $c$  if and only if  $gn_{s,d}(G) \leq K$ . Note that the reductions presented below are actually FPT-reduction and preserve approximation ratio. Therefore, since the SET COVER Problem is W[2]-hard (when parameterized by the size  $c$  of the set cover) and has no  $\alpha' \ln(n)$  approximation algorithm for some constant  $0 < \alpha' < 1$  (unless  $P=NP$ ) [3], we not only prove the NP-hardness but also the fact that the problem is W[2]-hard (when parameterized by the number of guards) and cannot be approximated in polynomial time up to some logarithmic ratio (unless  $P = NP$ ).

**Definition 4.** Given integers  $s \geq 2$  and  $d \geq 0$ , let  $p = p(s, d) = d + \left\lceil \frac{d+1}{s-1} \right\rceil$  and  $q = q(s, d)$  be

$$q(s, d) = \begin{cases} 0, & \text{if } d+1 < s < 2d+2, \\ d + \left\lceil \frac{d}{s-1} \right\rceil, & \text{if } s \leq d+1, \\ d, & \text{otherwise.} \end{cases}$$

Let  $(\mathcal{S}, c)$  be an instance of Set Cover, where  $\mathcal{S} = \{S_1, \dots, S_m\}$ , and let  $U = S_1 \cup \dots \cup S_m = \{u_1, \dots, u_n\}$ . Let  $K = K_{s,d}(\mathcal{S}, c)$  be:

$$K_{s,d}(\mathcal{S}, c) = \begin{cases} c, & \text{if } d+1 < s < 2d+2, \\ c+2, & \text{if } s \leq d+1 \text{ and } 1 \leq d \bmod (s-1) < \frac{s}{2} - 1, \\ c+1, & \text{otherwise.} \end{cases}$$

Let  $G = G_{s,d}(\mathcal{S}, c)$  be the graph defined as follows: for every set  $S_j \in \mathcal{S}$ , create a new vertex  $S_j$  in  $G$  and, for every element  $u_i \in U$ , create a path  $U_i$  with  $p$  vertices  $u_{i,1}, \dots, u_{i,p}$ . Make  $\{S_1, \dots, S_m\}$  be a clique in  $G$  (add all possible edges). If  $u_i \in S_j$ , add the edge  $u_{i,1}S_j$  in  $G$ . Create a new vertex  $z_0$  and add all possible edges between  $z_0$  and  $\{S_1, \dots, S_m\}$  in  $G$ . Finally, if  $q > 0$ , create a path  $Z$  with  $q$  vertices  $z_1, \dots, z_q$ , and add the edge  $z_0z_1$ . Moreover, if  $s \leq d+1$  and  $1 \leq d \bmod (s-1) < \frac{s}{2} - 1$ , then create a path  $Z'$  with  $q$  vertices  $z'_1, \dots, z'_q$  and add the edge  $z_0z'_1$ .

See Figures 1-3 for examples.

**Lemma 5.** *Given a graph  $G$  and an integer  $K > 0$ , deciding if  $gn_{s,d}(G) \leq K$  is NP-hard for every  $s, d \geq 0$  such that  $d+1 < s < 2d+2$ .*

*Proof.* Reduction from SET COVER. Let  $(\mathcal{S}, c)$  be an instance of Set Cover. Recall Definition 4 and let  $p = p(s, d) = d+1$ ,  $q = q(s, d) = 0$ ,  $G = G_{s,d}(\mathcal{S}, c)$  and  $K = K_{s,d}(\mathcal{S}, c) = c$ .

Firstly, suppose that there is no cover  $\mathcal{C}$  of  $U$  with at most  $c$  sets in  $\mathcal{S}$ . We prove that the spy wins against at most  $K = c$  guards. Precisely, the spy starts in  $z_0$  and can win in one step. Indeed, since there are at most  $K$  guards and there is no cover of  $U$  with  $c$  sets in  $\mathcal{S}$ , then there exists some  $1 \leq i \leq n$  such that there is no guard in  $N[U_i]$ . Thus, the spy goes to  $u_{i,p}$  in one step (note that the distance from  $z_0$  to  $u_{i,p}$  is  $p+1 = d+2 \leq s$ ). During the guards' step, no guard can reach a vertex of  $U_i$ , and so the spy remains at distance at least  $d$  from all guards. Therefore, the spy wins.

Now, suppose that there is a cover  $\mathcal{C} = \{S_{j_1}, \dots, S_{j_c}\}$  of  $U$  with  $c$  sets in  $\mathcal{S}$ . For ease of presentation, we prove that  $c = K$  guards win if they are placed first. By Claim 1, this is sufficient to prove that  $gn_{s,d}(G) \leq K$ . The strategy of the guards is as follows. Occupy initially

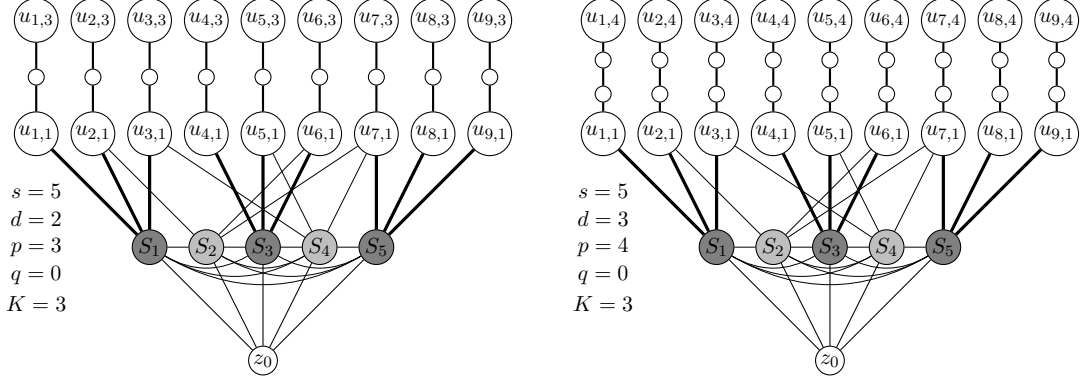


Figure 1: Reduction from Set Cover instance  $(\mathcal{S}, c)$ , where  $c = 3$ ,  $\mathcal{S} = \{S_1, S_2, S_3, S_4, S_5\}$ ,  $S_1 = \{1, 2, 3\}$ ,  $S_2 = \{2, 6, 7\}$ ,  $S_3 = \{4, 5, 6\}$ ,  $S_4 = \{3, 5, 7\}$ ,  $S_5 = \{7, 8, 9\}$  and  $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ . Cases for speed  $s = 5$  and distance  $d = 2, 3$ . Illustration of the proof of Lemma 5.

the vertices  $S_{j_1}, \dots, S_{j_c}$ . Since  $\mathcal{C}$  is a cover of  $U$ , we can define for any element  $u_i \in U$  an index  $c(i)$  such that  $u_i \in S_{c(i)} \in \mathcal{C}$ .

If the spy is not in  $\{u_{1,p}, \dots, u_{n,p}\}$ , then the guards occupy the initial vertices and then they control the spy. If the spy is in a vertex  $u_{i,p}$ , then the guard occupying  $S_{c(i)}$  goes to  $u_{i,1}$  and controls the spy. Since  $s < 2d + 2$ , the spy cannot go from  $u_{i,p}$  to other vertex  $u_{j,p}$  in one step ( $j \neq i$ ). Thus, if the spy leaves  $u_{i,p}$ , the guards reoccupy the initial vertices. With this strategy, the guards win the game.  $\square$

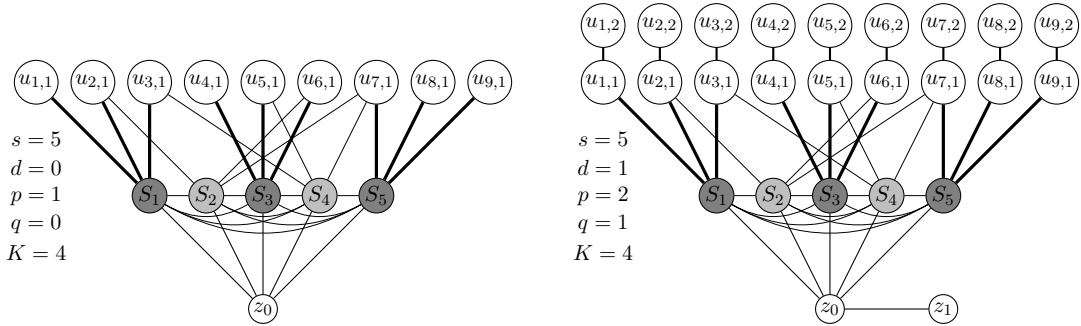


Figure 2: Reduction from Set Cover instance  $(\mathcal{S}, c)$ , where  $c = 3$ ,  $\mathcal{S} = \{S_1, S_2, S_3, S_4, S_5\}$ ,  $S_1 = \{1, 2, 3\}$ ,  $S_2 = \{2, 6, 7\}$ ,  $S_3 = \{4, 5, 6\}$ ,  $S_4 = \{3, 5, 7\}$ ,  $S_5 = \{7, 8, 9\}$  and  $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ . Cases for speed  $s = 5$  and distance  $d \in \{0, 1\}$ . Illustration of the proof of Lemma 6.

**Lemma 6.** *Given a graph  $G$  and an integer  $K$ , deciding if  $gn_{s,d}(G) \leq K$  is NP-hard for every  $s, d \geq 0$  such that  $s \geq 2d + 2$ .*

*Proof.* Reduction from SET COVER. Let  $(\mathcal{S}, c)$  be an instance of Set Cover. Recall Definition 4 and let  $p = p(s, d) = d + 1$ ,  $q = q(s, d) = d$ ,  $G = G_{s,d}(\mathcal{S}, c)$  and  $K = K_{s,d}(\mathcal{S}, c) = c + 1$ .

Firstly, suppose that there is no cover  $\mathcal{C}$  of  $U$  with at most  $c$  sets in  $\mathcal{S}$ . We prove that the spy wins against at most  $K = c + 1$  guards. Precisely, the spy starts in  $z_q$  and can win in one step.

Indeed, if initially no guard occupies a vertex in  $\{z_0, \dots, z_q\}$ , then the spy wins immediately. Therefore, let us assume that there is at least one guard in  $\{z_0, \dots, z_q\}$ . Since there are  $c + 1$  guards, then there is at most  $c$  guards outside  $\{z_0, \dots, z_q\}$ . Since there is no cover of  $U$  with  $c$  sets in  $\mathcal{S}$ , then there exists some  $1 \leq i \leq n$  such that there is no guard in  $N[U_i]$ . Thus, the spy goes to  $u_{i,p}$  in one step (note that the distance from  $z_q$  to  $u_{i,p}$  is  $p + q + 1 = 2d + 2 \leq s$ ) and wins since no guard can reach a vertex in  $U_i$  (i.e., no vertex at distance at most  $d$  from  $u_{i,p}$ ) during the next step.

Now, suppose that there is a cover  $\mathcal{C} = \{S_{j_1}, \dots, S_{j_c}\}$  of  $U$  with  $c$  sets in  $\mathcal{S}$ . For ease of presentation, we prove that  $c + 1 = K$  guards win if they are placed first. By Claim 1, this is sufficient to prove that  $gn_{s,d}(G) \leq K$ . The strategy of the guards is as follows. Occupy initially the vertices  $z_0, S_{j_1}, \dots, S_{j_c}$ . Since  $\mathcal{C}$  is a cover of  $U$ , we can define for any element  $u_i \in U$  an index  $c(i)$  such that  $u_i \in S_{c(i)} \in \mathcal{C}$ .

If the spy occupies a vertex not in  $\{u_{1,p}, \dots, u_{n,p}\}$ , then the guards keep their initial positions and control the spy. If the spy occupies the vertex  $u_{i,p}$ , then the guard occupying  $S_{c(i)}$  goes to  $u_{i,1}$  (controlling the spy) and the guard occupying  $z_0$  goes to  $S_{c(i)}$ . If the spy leaves  $u_{i,p}$  and occupies a vertex  $u_{j,p}$  with  $c(i) = c(j)$ , then the guard in  $S_{c(i)}$  goes to  $u_{j,1}$  (controlling the spy) and the guard in  $u_{i,1}$  goes to  $S_{c(i)}$ . If the spy leaves  $u_{i,p}$  and occupies a vertex  $u_{j,p}$  with  $c(i) \neq c(j)$ , then the guard occupying  $S_{c(j)}$  goes to  $u_{j,1}$  (controlling the spy), the guard in  $S_{c(i)}$  goes to  $S_{c(j)}$  and the guard in  $u_{i,1}$  goes to  $S_{c(i)}$ . If the spy leaves  $u_{i,p}$  to some vertex not in  $\{u_{1,p}, \dots, u_{n,p}\}$ , then the guards reoccupy the initial vertices: the guard in  $S_{c(i)}$  goes to  $z_0$  and the guard in  $u_{i,1}$  goes to  $S_{c(i)}$ . With this strategy, the guards win the game.  $\square$

Now consider the case  $d + 1 \geq s \geq 2$ . The next auxiliary lemma will be very useful.

**Lemma 7.** *Let  $s, d \geq 0$  be two integers such that  $d + 1 \geq s \geq 2$ , let  $p = p(s, d) = d + \left\lceil \frac{d+1}{s-1} \right\rceil$ ,  $q = q(s, d) = d + \left\lceil \frac{d}{s-1} \right\rceil$  and  $r = d \bmod (s - 1)$ . Note that  $p = q + 1$  if  $r = 0$  and  $p = q$  otherwise.*

*Let  $\ell \in \{p, q\}$ , let  $P = (x_{-1}, x_0, \dots, x_\ell)$  be a path and let us consider one guard playing the game in  $P$  against a spy with speed  $s$  and at distance  $d$ .*

- (a) *There is a winning strategy for the guard ensuring that the guard is in  $x_0$  when the spy occupies a vertex in  $\{x_{-1}, \dots, x_r\}$ ;*
- (b) *If  $r > 0$ , there are no winning strategies for the guard ensuring that it is in  $x_0$  when the spy is in  $x_j$  for  $j > r$ ;*
- (c) *If  $\ell = q$ , there are no winning strategies for the guard ensuring that it is in  $x_{-1}$  when the spy is in  $x_0$ .*
- (d) *If  $\ell = p$ , for every winning strategy for the guard, it must never occupy  $x_{-1}$ .*

*Proof.* (a). We first consider the case  $\ell = q$ . The strategy is defined as follows. If the spy occupies a vertex in  $\{x_{-1}, \dots, x_r\}$ , then the guard is at  $x_0$ . For any  $0 < j \leq \left\lceil \frac{d}{s-1} \right\rceil$ , if the spy occupies a vertex in  $\{x_{r+1+(j-1)s}, \dots, x_{r+js}\}$ , then the guard is at  $x_j$ . Note first that the strategy is well defined: for any move of the spy, the guard either stays idle or moves to a neighbor. Moreover, for any  $0 \leq j \leq \left\lceil \frac{d}{s-1} \right\rceil$ , the distance between them is  $r + j(s - 1)$ . While  $j \leq \left\lceil \frac{d}{s-1} \right\rceil$ , this distance is at most  $r + \left\lceil \frac{d}{s-1} \right\rceil (s - 1) = d$  (by definition of  $r$ ). It only remains to show that the strategy is defined for all possible positions of the spy. Note that the strategy is well defined when the spy occupies  $x_h$  for all  $h \leq r + \left\lceil \frac{d}{s-1} \right\rceil s$ . If  $r = 0$ , then  $r + \left\lceil \frac{d}{s-1} \right\rceil s = d + \left\lceil \frac{d}{s-1} \right\rceil = q = \ell$  and we are done (all positions have been considered). If  $r > 0$ , then  $r + \left\lceil \frac{d}{s-1} \right\rceil s = d + \left\lceil \frac{d}{s-1} \right\rceil - 1 = q - 1 = \ell - 1$ . Therefore, it only remains to define the strategy when the spy is in  $x_\ell$ , in which case, the guard occupies  $x_{1+\left\lfloor \frac{d}{s-1} \right\rfloor}$ .

Now, let us assume that  $\ell = p$ . Note that, if  $r > 0$ , then  $p = q$  and therefore, this case has already been treated. Hence, let us consider the case  $r = 0$ .

The strategy is defined as follows. If the spy is at  $x_{-1}$  or  $x_0$ , then the guard is at  $x_0$ . For any  $0 < j \leq \left\lfloor \frac{d}{s-1} \right\rfloor$ , if the spy occupies a vertex in  $\{x_{(j-1)s+1}, \dots, x_{js}\}$ , then the guard is at  $x_j$ . Since  $r = 0$ ,  $x_{\lfloor \frac{d}{s-1} \rfloor s} = x_q = x_{p-1} = x_{\ell-1}$ . Therefore, it only remains to define the position of the guard when the spy occupies  $x_\ell$ , in which case, the guard is at  $x_{\lfloor \frac{d}{s-1} \rfloor + 1}$ . Moreover, the distance between the spy and the guard is at most  $\ell - (\lfloor \frac{d}{s-1} \rfloor + 1) \leq d$ .

(b). If  $r > 0$  and the spy starts at  $x_{r+1}$ , then it goes at full speed toward  $x_\ell$ . After  $j = \left\lfloor \frac{d}{s-1} \right\rfloor$  steps, the spy occupies  $x_h$  for  $h = 1 + r + \left\lfloor \frac{d}{s-1} \right\rfloor s = \ell$  (as shown above when  $r > 0$ ), and the guard can only occupy a vertex in  $\{x_{-1}, \dots, x_j\}$ . Therefore, the distance between them is at least  $1 + r + \left\lfloor \frac{d}{s-1} \right\rfloor (s-1) = 1 + d$  and the spy wins.

(c). If  $r > 0$ , the spy first goes to  $x_{r+1}$  while the guard can only go to  $x_0$  and the result follows from the previous item. If  $r = 0$ , then the spy goes at full speed toward  $x_\ell$ . After  $j = \left\lfloor \frac{d}{s-1} \right\rfloor$  steps, the spy occupies  $x_h$  for  $h = \left\lfloor \frac{d}{s-1} \right\rfloor s = \ell$  (as shown in item (a)), and the guard can only occupy a vertex in  $\{x_{-1}, \dots, x_{j-1}\}$ . Therefore, the distance between them is at least  $1 + \left\lfloor \frac{d}{s-1} \right\rfloor (s-1) = 1 + d$  and the spy wins.

(d). Finally, assume that the spy starts in  $x_{-1}$  and goes at full speed to  $x_\ell$ . After  $j > 0$  steps, the spy occupies  $x_{js-1}$  and the guard occupies  $x_{j-1}$ . Therefore, the distance between them is  $j(s-1)$  which is at most  $d$  if and only if  $j \leq \left\lfloor \frac{d}{s-1} \right\rfloor$ . Let us set  $j_0 = \left\lfloor \frac{d}{s-1} \right\rfloor$  and note that  $sj_0 - 1 = s \left\lfloor \frac{d}{s-1} \right\rfloor - 1 = (s-1) \left\lfloor \frac{d}{s-1} \right\rfloor + \left\lfloor \frac{d}{s-1} \right\rfloor - 1 = d - r + \left\lfloor \frac{d}{s-1} \right\rfloor - 1 = d - r + \left\lfloor \frac{d+1}{s-1} \right\rfloor - 2 = p - 2 - r$ . After step  $j_0$ , the spy occupies  $x_{sj_0-1}$  and is at distance exactly  $d$  from the guard. During the step  $j_0+1$ , the spy can progress by at least two edges toward  $x_p$  (because  $s \geq 2$  and  $sj_0 - 1 \leq p-2$ ) while the guard can progress of at most one edge. Therefore, the distance between them is at least  $d+1$  and the spy wins.  $\square$

Now, let us consider the case when  $s \leq d+1$  and  $r = d \bmod (s-1) \geq \left\lceil \frac{s}{2} \right\rceil - 1$  or  $r = 0$ .

**Lemma 8.** *Given a graph  $G$  and an integer  $K$ , deciding if  $gn_{s,d}(G) \leq K$  is NP-hard for every  $s, d > 0$  such that  $2 \leq s \leq d+1$  and  $r = d \bmod (s-1) \in \{\left\lceil \frac{s}{2} \right\rceil - 1, \dots, s-2, 0\}$ .*

*Proof.* Reduction from SET COVER. Let  $(\mathcal{S}, c)$  be an instance of Set Cover. Recall Definition 4 and let  $p = p(s, d) = d + \left\lceil \frac{d+1}{s-1} \right\rceil$ ,  $q = q(s, d) = d + \left\lceil \frac{d}{s-1} \right\rceil$ ,  $r = d \bmod (s-1)$ ,  $G = G_{s,d}(\mathcal{S}, c)$  and  $K = K_{s,d}(\mathcal{S}, c) = c + 1$ .

Firstly, suppose that there is no cover  $\mathcal{C}$  of  $U$  with at most  $c$  sets in  $\mathcal{S}$ . We prove that the spy wins against at most  $K = c + 1$  guards. Precisely, the spy starts in  $z_0$  and can win as follows. If no guards are occupying a vertex in  $\{z_0, \dots, z_q\}$ , then by Lemma 7(c) the spy can move to  $z_q$  and win. Therefore, there must be a guard in  $\{z_0, \dots, z_q\}$  and so, at most  $c$  guards occupying vertices in  $V(G) \setminus \{z_0, \dots, z_q\}$ . Since there is no cover of  $U$  with at most  $c$  sets in  $\mathcal{S}$ , then there exists some  $1 \leq i \leq n$  such that there is no guard in  $N[U_i]$ . Thus, the spy goes at full speed  $s$  from  $z_0$  to  $u_{i,p}$ . The conditions are similar to the ones of Lemma 7(d) where the vertices in  $X = N(u_{1,p}) \setminus U_i$  (which are not occupied) play the role of  $x_0$ , and the vertices of  $N(X) \setminus N[U_i]$  (containing  $z_0$ ) play the role of  $x_{-1}$ . Therefore, the spy eventually wins.

Now, suppose that there is a cover  $\mathcal{C} = \{S_{j_1}, \dots, S_{j_c}\}$  of  $U$  with  $c$  sets in  $\mathcal{S}$ . In what follows, we describe a winning strategy for  $K = c + 1$  guards. The strategy of the guards will ensure that there is always a guard at every vertex of  $\mathcal{C}$ . Recall that, since  $\mathcal{C}$  is a cover of  $U$ , we can define for any element  $u_i \in U$  an index  $c(i)$  such that  $u_i \in S_{c(i)} \in \mathcal{C}$ .

The strategy is defined as follows.



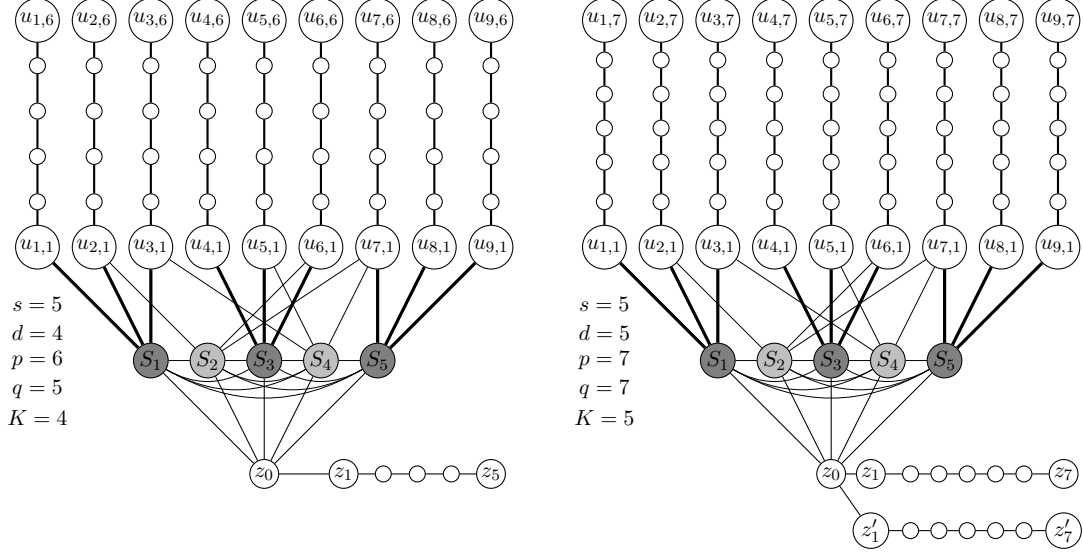


Figure 3: Reduction from Set Cover instance  $(\mathcal{S}, c)$ , where  $c = 3$ ,  $\mathcal{S} = \{S_1, S_2, S_3, S_4, S_5\}$ ,  $S_1 = \{1, 2, 3\}$ ,  $S_2 = \{2, 6, 7\}$ ,  $S_3 = \{4, 5, 6\}$ ,  $S_4 = \{3, 5, 7\}$ ,  $S_5 = \{7, 8, 9\}$  and  $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ . Cases for speed  $s = 5$  and distance  $d \in \{4, 5\}$ . Illustration of the proofs of Lemma 8 (left) and Lemma 9 (right).

- If the spy occupies a vertex in  $\{z_0, S_1, \dots, S_m\}$ , then the guards occupy the vertices in  $\{z_0, S_{j_1}, \dots, S_{j_c}\}$ .
- If the spy occupies a vertex in  $U_i$  for  $i \leq n$ , let  $P_i$  be the path induced by  $U_i$ ,  $S_{c(i)}$  and  $z_0$ . Let us apply Lemma 7(a) on  $P_i$  with  $\ell = p$ ,  $z_0$  plays the role of  $x_{-1}$  and  $S_{c(i)}$  plays the role of  $x_0$ . By Lemma 7(a), there exists a strategy allowing one guard to control the spy and such that the guard occupies  $S_{c(i)}$  if the spy occupies a vertex in  $\{z_0, S_{c(i)}, u_{i,1}, \dots, u_{i,r}\}$ . In that case, one guard, called the *follower*, follows the strategy defined by Lemma 7(a). The other guards occupy the vertices in  $\{S_{j_1}, \dots, S_{j_c}\}$  if the follower does not occupy  $S_{c(i)}$ , and they occupy  $\{z_0, S_{j_1}, \dots, S_{j_c}\} \setminus \{S_{c(i)}\}$  if the follower is at  $S_{c(i)}$ .
- If the spy occupies a vertex in  $Z$ , let  $Z'$  be the path induced by  $Z$ ,  $z_0$  and any vertex  $S_j$ . Let us apply Lemma 7(a) on  $Z'$  with  $\ell = q$ ,  $S_j$  plays the role of  $x_{-1}$  and  $z_0$  plays the role of  $x_0$ . By Lemma 7(a), there exists a strategy allowing one guard to control the spy and such that the guard occupies  $z_0$  if the spy occupies a vertex in  $\{z_0, u_{i,1}, \dots, u_{i,r}\}$  or any vertex  $S_j$ . In that case, one guard, called the *follower*, follows the strategy defined by Lemma 7(a). The other guards occupy the vertices in  $\{S_{j_1}, \dots, S_{j_c}\}$ .

For any position of the spy, the above strategy ensures that at least one guard controls the spy (by Lemma 7(a)). Hence, it only remains to prove that the strategy is valid, i.e., that, for any move of the spy, the guards can move accordingly. There are several cases to be considered.

- If the spy goes from a vertex in some  $U_i$  to another vertex of the same  $U_i$  or to a vertex in  $\{z_0, S_1, \dots, S_m\}$ . Then, the follower moves accordingly to the strategy of Lemma 7(a). If this move leads the follower to  $S_{c(i)}$  (in particular, by the property of the strategy of Lemma 7(a), it is the case if the spy reaches a vertex in  $\{z_0, S_1, \dots, S_m\}$ ), then the guard that was occupying  $S_{c(i)}$  goes to  $z_0$ . Therefore, all guards' moves are valid (if they move, they go to one of their neighbors).

By symmetry of the strategy (which is positional), the strategy of the guards is also valid if the spy moves from  $\{z_0, S_1, \dots, S_m\}$  to some  $U_i$ .

The case when the spy goes from a vertex of  $Z$  to  $Z$ , or from  $Z$  to  $\{z_0, S_1, \dots, S_m\}$  is similar.

- If the spy goes from a vertex in  $U_i$  to a vertex in  $U_j$  for some  $i \neq j$ . Note that, by the property of the strategy of Lemma 7(a), the follower has to be either in  $u_{i,1}$  or in  $S_{c(i)}$  after the spy's move (this is because, if the spy is able to go from  $U_i$  to  $U_j$ , it could also have gone to  $z_0$ , and the strategy ensures that, in that case, the follower must be able to reach  $S_{c(i)}$ ).

If the follower was in  $u_{i,1}$  (after the spy's move), then the guard at  $S_{c(j)}$  becomes the new follower (recall that all vertices in  $\{S_{j_1}, \dots, S_{j_c}\}$  are always occupied). If the strategy of the follower (in  $P_j$ ) asks it to move, the new follower moves (in which case, it goes to  $u_{j,1}$ ), then the guard at  $u_{i,1}$  goes to  $S_{c(i)}$ . Finally, if  $c(i) \neq c(j)$ , the guard that was occupying  $S_{c(i)}$  goes to  $S_{c(j)}$ . If the strategy of the follower is to stay idle, then the guard at  $u_{i,1}$  goes to  $S_{c(i)}$  and the guard that was at  $S_{c(i)}$  goes to  $z_0$ .

Otherwise, the follower was at  $S_{c(i)}$ , then the guards occupy  $\{z_0, S_{j_1}, \dots, S_{j_c}\}$ . In that case, the guard at  $S_{c(j)}$  becomes the new follower. If it has to move (to  $u_{j,1}$ ), then the guard at  $z_0$  replaces it at  $S_{c(j)}$ .

It is important to note that, in all cases, when the spy enters in  $U_j$ , the new follower was occupying  $S_{c(j)}$  (which plays the role of  $x_0$  in Lemma 7(a)), and therefore it can apply the strategy described in Lemma 7(a).

- The last case is when the spy goes from a vertex in  $U_i$  to a vertex in  $Z$ . If  $z_0$  was occupied by a guard then it becomes the follower and apply the strategy of Lemma 7(a). If  $z_0$  was not occupied, then it means that the guards were occupying the vertices in  $\{u_{1,i}, S_{j_1}, \dots, S_{j_c}\}$ . In particular, the follower was occupying  $u_{i,1}$  (because, by the property of the strategy of Lemma 7(a), this guard must be able to go to  $S_{c(i)}$  (i.e.,  $x_0$ ) when the spy can reach  $z_0$  (playing the role of  $x_{-1}$ ). Moreover, if the guard is occupying  $u_{i,1}$ , it must be because the spy was (before its last move) at  $u_{i,h}$  for  $h > r$  (otherwise, by the property of the strategy, the guard would be at  $S_{c(i)}$ ).

There are two cases depending whether  $r = 0$  or  $r \geq \lceil \frac{s}{2} \rceil - 1$  (the moves are the same, but the reason of their validity is different).

- If  $r = 0$ , note that  $p = q + 1$ . In that case, Lemma 7(a) can be applied on the path  $(u_{1,i}, S_{c(i)}, z_0, \dots, z_q)$  (playing the role respectively of  $(x_{-1}, x_0, x_1, \dots, x_p)$ ). Therefore, the guard at  $S_{c(i)}$  becomes the follower. It goes to  $z_0$  while the guard at  $u_{1,i}$  goes to  $S_{c(i)}$ .
- If  $r \geq \lceil \frac{s}{2} \rceil - 1$ , because the spy was at  $u_{i,h}$  for  $h > r$ , this implies that, after its move, the spy reaches a vertex  $z_q \in Z$  for  $q \leq r$ . In that case, the guard at  $S_{c(i)}$  goes to  $z_0$  and becomes the follower (this satisfies the conditions of the strategy of Lemma 7(a), because  $q \leq r$ ) and the guard at  $u_{1,i}$  goes to  $S_{c(i)}$ .

□

Finally, let us consider the case  $s \leq d + 1$  and  $1 \leq r = d \bmod (s - 1) < \frac{s}{2} - 1$ . Recall that, in this case, we have added another path  $Z'$  to  $G_{s,d}(\mathcal{S}, c)$ .

**Lemma 9.** *Given a graph  $G$  and an integer  $K$ , deciding if  $gn_{s,d}(G) \leq K$  is NP-hard for every  $s, d > 0$  such that  $s \leq d + 1$  and  $1 \leq r = d \bmod (s - 1) < \frac{s}{2} - 1$ .*

*Proof.* Reduction from SET COVER. Let  $(\mathcal{S}, c)$  be an instance of Set Cover. Recall Definition 4 and let  $p = p(s, d) = d + \lceil \frac{d+1}{s-1} \rceil$ ,  $q = q(s, d) = d + \lceil \frac{d}{s-1} \rceil$ ,  $G = G_{s,d}(\mathcal{S}, c)$  and  $K = K_{s,d}(\mathcal{S}, c) = c + 2$ . Notice that, since  $r = d \bmod (s - 1) \neq 0$ , then  $p = q$ .

Firstly, suppose that there is no cover  $\mathcal{C}$  of  $U$  with at most  $c$  sets in  $\mathcal{S}$ . We prove that the spy wins against at most  $K = c + 2$  guards. Precisely, the spy starts in  $z'_{r+1}$  and proceeds as follows. If no guards are occupying a vertex in  $\{z'_1, \dots, z'_q\}$ , then by Lemma 7(b) the spy can

move at full speed to  $z'_q$  and win. Moreover, if no guards are occupying a vertex in  $\{z_0, \dots, z_q\}$ , then, in one step, the spy goes to  $z_{r+1}$  (which is at distance  $2r + 2 < s$  by the assumption on  $r$ ) and, by Lemma 7(b), the spy will win by moving at full speed to  $z_q$ . Therefore, there must be at most  $c$  guards at the vertices in  $V(G) \setminus \{z_0, z_1, z'_1, \dots, z_q, z'_q\}$ . Since there is no cover of  $U$  with  $c$  sets in  $\mathcal{S}$ , then there exists some  $1 \leq i \leq n$  such that there is no guard in  $N[U_i]$ . Thus, in one step, the spy can go to  $u_{i,r+1}$  (at distance  $2r + 3 \leq s$  by the assumption on  $r$ ). From Lemma 7(b) the spy can move to  $u_{i,p}$  and wins.

Now, suppose that there is a cover  $\mathcal{C} = \{S_{j_1}, \dots, S_{j_c}\}$  of  $U$  with  $c$  sets in  $\mathcal{S}$ . In what follows, we describe a winning strategy for  $K = c + 2$  guards. Recall that, since  $\mathcal{C}$  is a cover of  $U$ , we can define for any element  $u_i \in U$  an index  $c(i)$  such that  $u_i \in S_{c(i)} \in \mathcal{C}$ . The strategy of the guards will ensure that there is always a guard at every vertex of  $\mathcal{C} \cup \{z_0\}$ . In addition, the last guard, called *follower*, follows the strategy described in Lemma 7(a) in one of the paths  $U_i$ , for  $1 \leq i \leq n$ ,  $Z$  or  $Z'$  depending on the position of the spy.

More precisely, if the spy is occupying a vertex in  $\{z_0, S_1, \dots, S_m\}$ , the guards occupy the vertices  $z_0, z_0, S_{j_1}, \dots, S_{j_c}$  (two guards in  $z_0$ ). When the spy arrives at a vertex in  $U_i$  for some  $i \leq n$  (resp., in  $Z$  or  $Z'$ ), the guard at  $S_{c(i)}$  (resp., at  $z_0$ ) plays the role of the follower in the corresponding path. The other  $c + 1$  guards reorganize themselves to occupy the vertices  $z_0, S_{j_1}, \dots, S_{j_c}$ .

In particular, when the spy goes from one path  $U_i$  (resp.,  $Z$ , resp.,  $Z'$ ) to another path  $U_j$  or  $Z$  or  $Z'$ , Lemma 7(a) ensures that the previous follower was either at  $u_{i,1}$  of  $S_{c(i)}$  (resp.,  $z_1$  or  $z_0$ , resp.,  $z'_1$  or  $z_0$ ). Therefore, it is possible for the guards (which are not the new follower) to reorganize themselves to occupy the vertices  $z_0, S_{j_1}, \dots, S_{j_c}$ .

The details are similar to the ones provided in the proof of Lemma 8 and are omitted.  $\square$

## 2.2 PSPACE-hardness in the directed case

Then, we consider a variant of our game played on digraphs. In this variant, both the guards and the spy can move only by following the orientation of the arcs. However, the distances are the ones of the underlying undirected graph. Moreover, in this section, we consider the variant when the guards are placed first. Recall that, in this setting, the goal of the guards is to control the spy after a finite number of steps. Given a digraph  $D$  and two integers  $d, s \in \mathbb{N}$ ,  $s > 0$ , let  $\tilde{g}n_{s,d}(D)$  denote the minimum number of guards required to control a spy with speed  $s$  at distance  $d$  in the digraph  $D$  and in this variant.

**Theorem 10.** *The problem of computing  $\tilde{g}n_{s,2}$  is PSPACE-hard in the class of DAGs.*

The result is obtained by reducing the PSPACE-complete Quantified Boolean Formula in Conjunctive Normal Form (QBF) problem. Given a set of boolean variables  $x_1, \dots, x_n$  and a boolean formula  $F = C_1 \wedge C_2 \wedge \dots \wedge C_m$  where  $C_j$  is a disjunction of literals, the QBF problem asks whether the expression  $\phi = Q_1 x_1 Q_2 x_2 \dots Q_n x_n F$  is true, where every  $Q_i$  is either  $\forall$  or  $\exists$ .

*Proof.* For ease of readability, the proof below is given for  $d = 2$  but can easily be adapted for any distance  $d$ .

Let  $\phi$  be a quantified boolean formula with  $n$  boolean variables. We construct a DAG  $D_\phi$  such that  $\phi$  is true if and only if  $n$  guards control a spy at distance 2 in  $D_\phi$  after a finite number of turns.

For each  $Q_i x_i$  of  $\phi$  we construct a gadget digraph  $D_i$ . If  $Q_i = \exists$  then  $V(D_i) = \{w_{i-1}, z_i^1, z_i^2, z_i^3, z_i^4, x_i, x_i^*, \bar{x}_i, \bar{x}_i^*, y_i, v_i, v_i', w_i\}$ , the arcs between the vertices are shown in figure 4a. If  $Q_i = \forall$  then  $V(D_i) = \{w_{i-1}, z_i^1, z_i^2, z_i^3, z_i^4, x_i, x_i^*, \bar{x}_i, \bar{x}_i^*, y_i, \bar{y}_i, v_i, \bar{v}_i, v_i', w_i\}$ , the arcs between the vertices are shown in figure 4b.

Observe that the vertex  $w_i$  appears in both  $D_i$  and  $D_{i+1}$ . It remains to establish a relationship between each clause and the variables it contains. For each clause  $C_i$  we create a vertex  $c_i$

in  $D_\phi$  and add an arc from  $w_n$  to  $c_i$ . We also add an arc from  $c_i$  to  $x_i(\bar{x}_i)$  if clause  $C_i$  contains the literal  $x_i(\bar{x}_i)$ .

An example of the digraph  $D_\phi$  for  $\phi = \exists x_1 \forall x_2 (x_1 \vee \bar{x}_2) \wedge (\bar{x}_1 \vee x_2)$  is shown on figure 4c.

It remains to prove that  $\phi$  is true if and only if  $g\bar{n}_{s,2}(D_\phi) = n$ .

First note that, for each gadget  $D_i$ , at least one guard has to pick a vertex from  $S_i = \{z_i^1, z_i^2, z_i^3\}$  as his initial position, otherwise the spy would pick  $z_i^1$  as his initial position and no guard could ever reach distance 2 from such a vertex, therefore the spy would win. We will refer to the guard initially in  $S_i$  as  $p_i$ . Since  $D_\phi$  has  $n$  such gadgets, then  $\bar{g}_2(D_\phi) \geq n$ . Furthermore, assuming that each guard  $p_i$  starts on  $z_i^1$  he can only occupy the vertices on the set  $R_i = \{z_i^1, z_i^2, z_i^3, z_i^4, x_i, x_i^*, \bar{x}_i, \bar{x}_i^*\}$  during the rest of the game.

Suppose that  $\phi = false$ . We describe a winning strategy for the spy playing against  $n$  guards. Let us assume that there is exactly one guard in each set  $S_i$ , that is, the spy cannot win by just initially positioning himself in one unprotected  $z_i^1$ . The spy starts on the vertex  $w_0$ .

Now, suppose that the spy is in some  $w_{i-1}$  of  $D_i(\forall)$ , then the only guard that can reach a vertex at distance at most 2 from  $w_{i-1}$  is  $p_i$  when he occupies the vertex  $z_i^4$ . The spy waits until the guard  $p_i$  moves to  $z_i^4$ , if the guard never does so the spy stays on  $w_{i-1}$  and wins the game. Therefore, suppose that  $p_i$  eventually moves to  $z_i^4$ , then the spy chooses between moving to  $y_i$  or  $\bar{y}_i$ , depending on the choice of the spy, the guard  $p_i$  is then forced to move to  $x_i^*$  or to  $\bar{x}_i^*$ , because these are the only vertices that are reachable for any guards that are at distance at most 2 from  $y_i$  and  $\bar{y}_i$  respectively. If  $p_i$  moves to  $x_i^*$  the corresponding variable  $x_i$  is set to *true*. Otherwise, if  $p_i$  moves to  $\bar{x}_i^*$  then  $x_i = false$ . It means that for a quantified variable  $\forall x_i$  the spy chooses the value of  $x_i$ .

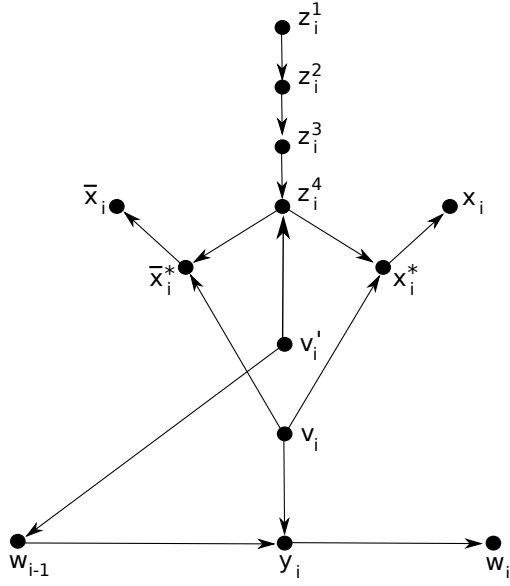
If the spy is in some  $w_{i-1}$  of  $D_i(\exists)$ , again, the only guard that can reach a vertex at distance at most 2 from the spy is  $p_i$  when he occupies the vertex  $z_i^4$ . The spy then waits until the guard  $p_i$  moves to  $z_i^4$  and then moves to  $y_i$ , this time  $p_i$  is not forced to move to specifically  $x_i^*$  or to  $\bar{x}_i^*$ , but he still must choose one of them. Again, if  $p_i$  moves to  $x_i^*$  the corresponding variable  $x_i$  is set to *true*, otherwise, if  $p_i$  moves to  $\bar{x}_i^*$  then  $x_i = false$ . It means that for a quantified variable  $\exists x_i$  the guards choose the value of  $x_i$ .

When  $p_n$  moves to  $x_n^*$  or  $\bar{x}_n^*$  each guard is on  $x_i^*(\bar{x}_i^*)$  or  $x_i(\bar{x}_i)$ . Observe that each guard can only reach a safe distance from the vertices  $c_j$  corresponding to the clauses that contains the literal he set true. Since  $\phi = false$  then the spy can choose between  $y_i$  and  $\bar{y}_i$  on gadgets  $D_i(\forall)$  in such a way that no matter how the guards choose  $x_i^*$  or  $\bar{x}_i^*$  on gadgets  $D_i(\exists)$  there is at least one vertex  $c_j$  that cannot be protected by any guard. Then the spy moves to such a vertex, stays there and wins the game.

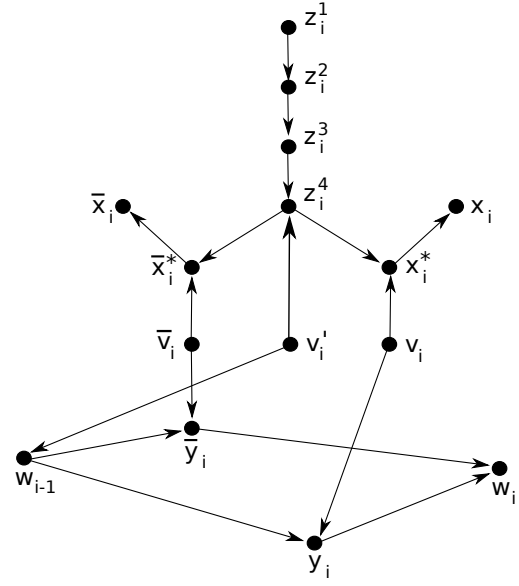
Suppose that  $\phi = true$ . Each guard  $p_i$ ,  $i = 1, \dots, n$ , will choose  $z_i^3$  as his initial position. If the spy choose as his initial position  $z_i^1, z_i^2, z_i^3, z_i^4, x_i^*$  or  $\bar{x}_i^*$  the guard  $p_i$  does not need to move since the spy is at distance at most 2 from  $z_i^3$ . The only vertices that the spy can go to from these initial positions that are not under the protection of  $p_i$  are  $x_i$  or  $\bar{x}_i$ . If he goes to any of them the guard  $p_i$  just moves to  $z_i^4$ . Since the spy cannot move anymore and is at distance at most 2 from a guard, the guards win the game. If the spy starts on some  $v_i, \bar{v}_i$  or  $v_i'$  then  $p_i$  moves to  $z_i^4$ , after that, if the spy goes to  $x_i^*, \bar{x}_i^*$  or  $z_i^4$  then  $p_i$  follows the same strategy from above. Therefore, the spy, independent of his initial position, must eventually move to a vertex  $w_i, y_i, \bar{y}_i$  or some clause vertex  $c_j$ , otherwise he loses.

Suppose that the spy is in some vertex  $w_{i-1}$  of  $D_i(\forall)$  then the guard  $p_i$  moves to  $z_i^4$  and controls the spy. The spy must move to  $y_i$  or  $\bar{y}_i$  forcing  $p_i$  to move to  $x_i^*$  or  $\bar{x}_i^*$  accordingly. Again, for a quantified variable  $\forall x_i$  the spy chooses the value of  $x_i$ . After the spy moves from  $y_i(\bar{y}_i)$  the cop moves to  $x_i(\bar{x}_i)$  and stays there forever.

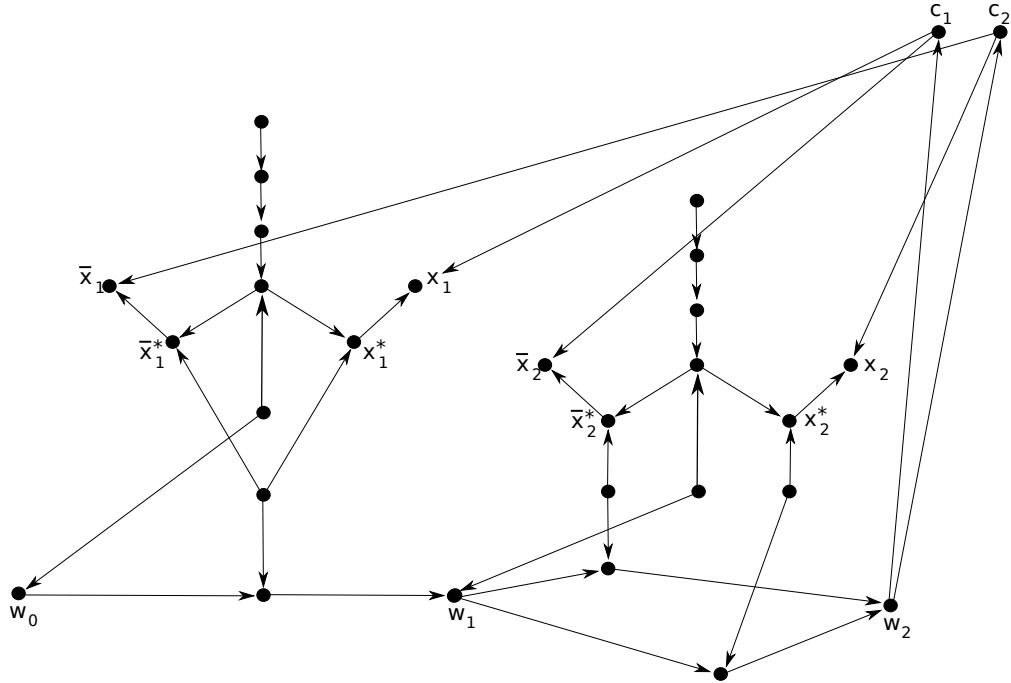
Similarly, if the spy is in some vertex  $w_{i-1}$  of  $D_i(\exists)$  then the guard  $p_i$  moves to  $z_i^4$  and controls the spy. The spy must move to  $y_i$ , this time  $p_i$  is not forced to move to specifically  $x_i^*$  or to  $\bar{x}_i^*$ , but he still must choose one of them. Therefore, for a quantified variable  $\exists x_i$  the guards choose the value of  $x_i$ . After the spy moves from  $y_i$  the cop moves to  $x_i$  or  $\bar{x}_i$  depending



(a) gadget  $D_i(\exists)$  for existential quantifier.



(b) Gadget  $D_i(\forall)$  for universal quantifier.



(c) Example of the graph  $D_\phi$  for the formula  $\phi = \exists x_1 \forall x_2 (x_1 \vee \bar{x}_2) \wedge (\bar{x}_1 \vee x_2)$ .

Figure 4

of his previous movement and stays on that vertex forever.

Observe that after the spy moves from  $y_n$  or  $\bar{y}_n$  every guard is at distance 2 from  $w_n$ , at distance 1 from each clause vertex that contains the literal he chose to set true and at distance 2 from each of the other literals of these clauses. Since  $\phi = \text{true}$  then the guards can choose between  $y_i$  and  $\bar{y}_i$  on gadgets  $D_i(\exists)$  in such a way that no matter how the spy chooses  $x_i^*$  or  $\bar{x}_i^*$  on gadgets  $D_i(\forall)$  all clause vertices are at distance 1 from at least one guard. Therefore the only vertices reachable for the spy are at distance at most 2 from the guards.  $\square$

Note that, although the construction above works for  $d = 2$ , it could be adapted to any  $d \geq 2$  by making the paths between certain pair of vertices have length  $d$  and adjusting the directions of some arcs.

The question of the complexity of the spy game in undirected graphs is left open. Is it PSPACE-hard, or more probably EXPTIME-complete as Cops and Robber games [15]? The question of parameterized complexity is also open.

### 3 Case of paths and rings

In this section, we characterize optimal strategies in the case of two simple topologies.

#### 3.1 Paths

First, let us consider the case of paths.

**Theorem 11.** *Let  $s > 1$  and  $d \geq 0$ . Let  $P = (v_0, \dots, v_{n-1})$  be any  $n$ -node path.*

$$gn_{s,d}(P) = \left\lceil \frac{n}{2d+2 + \left\lfloor \frac{2d}{s-1} \right\rfloor} \right\rceil$$

*Proof.* Let us set  $2d = q(s-1) + r$  where  $q = \left\lfloor \frac{2d}{s-1} \right\rfloor$  and  $r < s-1$  (note that, if  $s > 2d+1$ , then  $q = 0$  and  $r = 2d$ ). Note also that  $2d+2+q = qs+r+2$

Let us first show that the spy can win against at most  $\left\lceil \frac{n}{2d+2 + \left\lfloor \frac{2d}{s-1} \right\rfloor} \right\rceil - 1$  guards. The spy starts in  $v_0$ , so there must be a guard, called Guard 1, at some vertex in  $\{v_0, \dots, v_d\}$  to control the spy. Then, in  $q$  steps, the spy goes to  $v_{qs}$  while Guard 1 can only reach a vertex in  $\{v_0, \dots, v_{d+q}\}$ . Note that the distance between the spy and Guard 1 is then at least  $qs - (d+q) = d-r$ . During the next step  $q+1$ , the spy reaches vertex  $v_{qs+r+2}$  (note that it is possible since  $r+2 \leq s$ ). Guard 1 can only go to  $v_{d+q+1}$  and therefore it is at distance at least  $d+1$  from the spy. Therefore, there must be another guard, called Guard 2, occupying a vertex in  $\{v_{qs+r+2-d}, \dots, v_{qs+r+2+d}\}$  to control the spy. Going on this way, for  $0 < j < \left\lceil \frac{n}{2d+2 + \left\lfloor \frac{2d}{s-1} \right\rfloor} \right\rceil - 1$ , after  $j(q+1)$  turns, the spy occupies vertex  $v_{j(qs+r+2)}$  and there must be a guard, called Guard  $j+1$ , occupying some vertex in  $\{v_{j(qs+r+2)-d}, \dots, v_{j(qs+r+2)+d}\}$ . Moreover, all the  $j$  previous guards (Guard 1 to Guard  $j$ ) are occupying some vertices in  $\{v_0, \dots, v_{j(qs+r+2)-d-1}\}$ . In particular, just after  $j_0(q+1)$  turns, where  $j_0 = \left\lceil \frac{n}{2d+2 + \left\lfloor \frac{2d}{s-1} \right\rfloor} \right\rceil - 2$ , all the  $\left\lceil \frac{n}{2d+2 + \left\lfloor \frac{2d}{s-1} \right\rfloor} \right\rceil - 1$  guards are occupying vertices in  $\{v_0, \dots, v_{j_0(qs+r+2)+d}\}$  while the spy is at  $v_{j_0(qs+r+2)}$ . Therefore, during the next  $q+1$  turns, the spy goes to  $v_{(j_0+1)(qs+r+2)}$ . Note that  $(j_0+1)(qs+r+2) = \left( \left\lceil \frac{n}{2d+2 + \left\lfloor \frac{2d}{s-1} \right\rfloor} \right\rceil - 1 \right)(qs+r+2) = \left( \left\lceil \frac{n}{2d+2+q} \right\rceil - 1 \right)(2d+2+q) < n$ ,

so the move is possible. During these last  $q + 1$  steps, all guards can only reach vertices in  $\{v_0, \dots, v_{j_0(qs+r+2)+d+q+1}\}$  and, therefore, are all at distance at least  $d + 1$  from the spy (indeed,  $(j_0 + 1)(qs + r + 2) - (j_0(qs + r + 2) + d + q + 1) = d + 1$ ). Hence, the spy wins.

Finally, let us describe a winning strategy for  $\left\lceil \frac{n}{2d+2+\lfloor \frac{2d}{s-1} \rfloor} \right\rceil$  guards. For  $0 \leq j < \left\lceil \frac{n}{2d+2+\lfloor \frac{2d}{s-1} \rfloor} \right\rceil - 1$ , let  $P_j = (v_{j(qs+r+2)}, \dots, v_{(j+1)(qs+r+2)-1})$ . Moreover, for  $j_0 = \left\lceil \frac{n}{2d+2+\lfloor \frac{2d}{s-1} \rfloor} \right\rceil - 1$ , let  $P_{j_0} = (v_{j_0(qs+r+2)}, \dots, v_{n-1})$  (note that  $n-1 \leq (j_0+1)(qs+r+2)-1$ ). The strategy simply uses one guard, called Guard  $j$ , for each subpath  $P_j$ . Precisely, for any  $0 \leq j \leq \left\lceil \frac{n}{2d+2+\lfloor \frac{2d}{s-1} \rfloor} \right\rceil - 1$ ,

- for any  $0 \leq h < q$ , if the spy occupies a vertex in  $\{v_{j(qs+r+2)+hs+1}, \dots, v_{j(qs+r+2)+(h+1)s}\}$  then Guard  $j$  occupies  $v_{j(qs+r+2)+d+h+1}$ ;
- if the spy occupies a vertex in  $\{v_{j(qs+r+2)+qs+1}, \dots, v_{j(qs+r+2)+qs+r+1}\}$ , then Guard  $j$  occupies  $v_{j(qs+r+2)+d+q+1}$ ;
- if the spy occupies  $v_{j(qs+r+2)}$  or some subpath  $P_i$ , with  $i < j$ , then Guard  $j$  occupies  $v_{j(qs+r+2)+d}$ ;
- Finally, if the spy occupies some subpath  $P_i$ , with  $i > j$ , then Guard  $j$  occupies  $v_{j(qs+r+2)+q+1+d}$ .

It can be checked that, following this strategy, the guards always control the spy. Moreover, for any move of the spy, the guards can move accordingly to this strategy.  $\square$

### 3.2 Cycles

We then consider the case of cycles. Let us first start with the case  $2d < s - 1$ .

**Lemma 12.** *Let  $0 \leq 2d < s - 1$ . For any cycle  $C_n$  with  $n$  vertices,*

$$gn_{s,d}(C_n) \leq \left\lceil \frac{n}{2d+3} \right\rceil.$$

*Proof.* Since the number of guards cannot decrease when  $n$  increases, we may assume that  $\frac{n}{2d+3} = k \in \mathbb{N}$ . Let  $C_n = (v_0, \dots, v_{n-1})$ . Let us describe a strategy using  $k$  guards.

Assume that the spy is initially in  $v_0$ . The guards are placed at vertices  $v_{d+j(2d+3)}$ , for any  $0 \leq j < k$ . Note that, in particular, the last guard is placed at  $v_{d+(k-1)(2d+3)} = v_{n-d-3}$  since  $n = (2d+3)k$ .

Now, the guards are at distance at most  $d$  from all vertices but the vertices  $v_{2d+1+j(2d+3)}$ ,  $v_{2d+2+j(2d+3)}$  for any  $0 \leq j < k$ . If the spy goes to  $v_{2d+1+j(2d+3)}$  for some  $0 \leq j < k$ , then all guards move clockwise. If the spy goes to  $v_{2d+2+j(2d+3)}$  for some  $0 \leq j < k$ , then all guards move counter-clockwise. Both cases are symmetric to the initial one. In any other case, the guards do not move. Clearly, such a strategy can perpetually ensure that at least one guard controls the spy at distance  $d$ .  $\square$

**Lemma 13.** *Let  $s > 1$  odd and  $d > 0$  be two integers such that  $\frac{2d}{s-1} \in \mathbb{N}^*$ . For any cycle  $C_n$  with  $n$  nodes,*

$$gn_{s,d}(C_n) \leq \left\lceil \frac{n + 2(\frac{2d}{s-1})}{2(d + \frac{2d}{s-1})} \right\rceil.$$

*Proof.* Let us set  $2d = q(s-1)$  where  $q = \frac{2d}{s-1} \in \mathbb{N}^*$ , and let  $X = 2(d + \frac{2d}{s-1}) = q(s+1)$ .

Since the number of guards cannot decrease when  $n$  increases, we may assume that  $\frac{n+2(\frac{2d}{s-1})}{2(d+\frac{2d}{s-1})} = \frac{n+2q}{X} = k \in \mathbb{N}$ . Let us describe a strategy using  $k$  guards.

Let  $v_0$  be the initial position of the spy, and the cycle is  $(v_0, \dots, v_{n-1})$ . The guards are placed at vertices  $v_{d+jX}$ , for any  $0 \leq j < k$ . Let us call the guard at  $v_{d+jX}$  as the Guard  $j$ , for any  $0 \leq j < k$ . Note that, in particular, the Guard  $k-1$  is placed at  $v_{d+(k-1)X} = v_{n-d}$  since  $n-2d = (k-1)X$ . Therefore, the initial configuration is symmetric according to the symmetry axis passing through  $v_0$ .

- If the spy goes from  $v_0$  to any vertex in  $\{v_0, \dots, v_{\lfloor s/2 \rfloor}\}$ , no guards move. Note that Guard 0 still controls the spy.
- If the spy goes from  $\{v_0, \dots, v_{\lfloor s/2 \rfloor}\}$  to a vertex in  $\{v_{\lfloor s/2 \rfloor+1}, \dots, v_{\lfloor s/2 \rfloor+s}\}$ , then Guard 0 also goes clockwise to  $v_{d+1}$ . All other guards go counter-clockwise to  $v_{d+jX-1}$ , for every  $0 < j < k$ .
- For  $0 < h < q$ , when the spy goes from a vertex in  $\{v_{\lfloor s/2 \rfloor+(h-1)s+1}, \dots, v_{\lfloor s/2 \rfloor+hs}\}$  to a vertex in  $\{v_{\lfloor s/2 \rfloor+hs+1}, \dots, v_{\lfloor s/2 \rfloor+(h+1)s}\}$ , then Guard 0 also goes clockwise to  $v_{d+h+1}$ . Note that Guard 0 is at distance at most  $d$  from the spy. All other guards go counter-clockwise to  $v_{d+jX-h-1}$ , for every  $0 < j < k$ .
- For  $1 \leq h \leq q$ , when the spy goes from a vertex in  $\{v_{\lfloor s/2 \rfloor+(h-1)s+1}, \dots, v_{\lfloor s/2 \rfloor+hs}\}$  to a vertex in  $\{v_{\lfloor s/2 \rfloor+(h-1)s+1}, \dots, v_{\lfloor s/2 \rfloor+hs}\}$ , no guards move.

The following remarks show that the rules above fully describe the strategy of  $k$  guards. That is, the behaviour of the guards according to any spy's move can be derived from above rules by symmetry.

First, all previous moves are reversible. For instance, if the spy goes from  $\{v_{\lfloor s/2 \rfloor+1+hs}, \dots, v_{\lfloor s/2 \rfloor+(h+1)s}\}$  to  $\{v_{\lfloor s/2 \rfloor+1+(h-1)s}, \dots, v_{\lfloor s/2 \rfloor+hs}\}$  (for  $1 \leq h < q$ ), then Guard 0 goes back to  $v_{d+h}$ , and all other guards go back to  $v_{d+jX-h}$ , for every  $0 < j < k$ .

Second, let us consider the configuration when the spy arrives in  $\{v_{qs}, \dots, v_{\lfloor s/2 \rfloor+qs}\}$ . At this step, for any  $0 < j < k$ , Guard  $j$  is occupying  $v_{d+jX-q} = v_{d+(j-1)X+X-q}$ . Since  $X-q = q(s+1)-q = qs$ , this means that, for any  $0 < j < k$ , Guard  $j$  is occupying vertex  $v_{qs+d+(j-1)X}$ . Moreover, Guard 0 is occupying vertex  $v_{d+q} = v_{qs-d}$ . Therefore, the situation is symmetric to the initial one up to a rotation (where  $v_{qs}$  replaces  $v_0$ , Guard 0 becomes Guard  $k-1$ , and Guard  $j$  becomes Guard  $j-1$  for every  $0 < j < k$ ).  $\square$

**Corollary 14.** *Let  $2d \geq s-1 > 0$ . Let  $2d = q(s-1) + r$  ( $0 \leq r < s-1$ ) and  $2d = q's + r'$  ( $0 \leq r' < s$ ). Let  $(q^*, r^*) = (q, r)$  if  $s$  odd and  $(q^*, r^*) = (q', r')$  otherwise.*

*For any cycle  $C_n$  with  $n$  nodes,*

$$gn_{s,d}(C_n) \leq \left\lceil \frac{n+2q^*}{2(d+q^*)-r^*} \right\rceil.$$

*Proof.* First, let us assume that  $s$  is odd. Let  $d' = d - r/2$  (note that  $r$  has to be even since  $s$  is odd). By Lemma 13,  $gn_{s,d'}(C_n) \leq \left\lceil \frac{n+2q}{2(d'+q)} \right\rceil$ . Then, the result follows since  $gn_{s,d}(C_n) \leq gn_{s,d'}(C_n)$ .

If  $s$  is even, we use the fact that  $gn_{s,d}(C_n) \leq gn_{s+1,d}(C_n)$  and apply the result above to show that  $gn_{s+1,d}(C_n) \leq \left\lceil \frac{n+2q'}{2(d+q')-r'} \right\rceil$ .  $\square$

**Lemma 15.** *Let  $s > 1$  and  $d \geq 0$ . Let  $C_n = (v_0, \dots, v_{n-1})$  be any  $n$ -node cycle.*

$$gn_{s,d}(C_n) \geq \left\lceil \frac{n+2 \left\lfloor \frac{2d}{s-1} \right\rfloor}{2(d + \left\lfloor \frac{2d}{s-1} \right\rfloor) + 3} \right\rceil$$

*Proof.* Let us set  $2d = q(s-1) + r$  where  $q = \left\lfloor \frac{2d}{s-1} \right\rfloor$  and  $r < s-1$  (note that, if  $s > 2d+1$ , then  $q = 0$  and  $r = 2d$ ). Note also that  $2d+2+q = qs+r+2$ . All integers below must be understood modulo  $n$ .



Let us show that the spy can win against a team of  $X < \left\lceil \frac{n+2q}{2(d+q)+3} \right\rceil$  guards. If the spy starts in  $v_0$ , there must be a guard, called Guard 1, at some vertex in  $\{v_{n-d}, \dots, v_0, \dots, v_d\}$  to control the spy. Since the spy's speed is greater than the guards' speed, the spy can move clockwise so that he reaches a vertex that is distance  $d+2$  from Guard 1 in a finite number of turns (before the guards' turn). Thus, after the guard turn, we may set  $v_0$  (up to renaming the vertices) to be the new position of the spy and so, Guard 1 is at a vertex in  $\{v_{n-d-3}, v_{n-d-2}, v_{n-d-1}\}$ .

Since Guard 1 is at distance at least  $d+1$  from the spy, there must be another guard, called Guard 2, occupying a vertex in  $\{v_{n-d}, \dots, v_0, \dots, v_d\}$  to control the spy. The spy goes at full speed clockwise and Guard 1 may go at full speed counterclockwise.

Then, after step  $q$ , the spy occupies  $v_{qs}$  while Guard 2 occupies a vertex in  $\{v_{qs-d}, \dots, v_{d+q}\}$ . During the next step (Step  $q+1$ ) the spy goes to  $v_{qs+r+2}$  (note that it is possible since  $r+2 \leq s$ ). In this case, Guard 2 can only go to a vertex in  $\{v_{qs-d-1}, \dots, v_{d+q+1}\}$  and therefore it is at distance at least  $d+1$  from the spy and cannot control it anymore.

Therefore, there must be another guard, called Guard 3, occupying a vertex in  $\{v_{qs+r+2-d}, \dots, v_{qs+r+2+d}\}$  to control the spy. Going on this way after  $(X-1)(q+1)$  steps, the spy is at  $v_{(X-1)(qs+r+2)} = v_\alpha$  while there are  $X$  guards occupying vertices in

$$\{v_{n-(d+3+(X-1)(q+1))}, \dots, v_{(X-2)(qs+r+2)+d+q+1}\} = \{v_\beta, \dots, v_\gamma\}.$$

Note that  $\alpha - \gamma = qs + r + 2 - d - q - 1 = 2d + 2 + q - d - q - 1 = d + 1$ . Therefore, the distance between the spy and  $v_\gamma$  is at least  $d+1$  and the spy can only be controlled from a guard in  $v_\beta$ . The distance between  $v_\beta$  and  $v_\alpha$  is:

$$n - (d + 3 + (X - 1)(q + 1) + (X - 1)(qs + r + 2)) = n + d + 2q - X(2q + 2d + 3).$$

Moreover,  $n + d + 2q - X(2q + 2d + 3) > d$  if and only if  $\frac{n+2q}{2(d+q)+3} > X$ .

Therefore, the distance between  $v_\alpha$  and  $v_\beta$  is at least  $d+1$  for  $X < \left\lceil \frac{n+2q}{2(d+q)+3} \right\rceil$  since  $X$  is an integer and thus, no guard controls the spy which wins.  $\square$

The above lemmas can be summarized with the following theorem.

**Theorem 16.** *Let  $s > 1$  and  $d \geq 0$  be two integers. For any cycle  $C_n$  with  $n$  nodes,*

- $gn_{s,d}(C_n) = \left\lceil \frac{n}{2d+3} \right\rceil$  if  $0 \leq 2d < s - 1$ ;
- If  $2d \geq s - 1$ , let  $2d = q(s - 1) + r$  ( $0 \leq r < s - 1$ ) and  $2d = q's + r'$  ( $0 \leq r' < s$ ). Let  $(q^*, r^*) = (q, r)$  if  $s$  odd and  $(q^*, r^*) = (q', r')$  otherwise. Then,

$$\left\lceil \frac{n + 2q}{2(d + q) + 3} \right\rceil \leq gn_{s,d}(C_n) \leq \left\lceil \frac{n + 2q^*}{2(d + q^*) - r^*} \right\rceil.$$

## 4 A variant of the Spy-game with a countdown

In this last section, we initiate the study of another interesting variant of the Spy game. Let  $s \geq 2$  and  $d$  be two integers representing the speed of the spy and the distance of control as previously. Let  $T$  be any integer, called the *horizon* of the game. The spy and the guards are initially placed (in this order) at some vertices of a graph and then, turn-by-turn move at speed  $s$  and 1 respectively. The objective of the guards is now to ensure that at least one guard is at distance at most  $d$  from the spy after the  $T^{th}$  turn. The spy wins otherwise. That is, contrary to the Spy-game, the game terminates after the  $T^{th}$  turn, and there are no other constraints (except the speed) on the positions of the guards before this last turn.

Let us define  $g_T^{s,d}(G)$  as the minimum number of guards required to win this game against any strategy of a spy on a graph  $G$ . The rest of this section only covers the case of paths and starts with the following observation.

**Observation 17.** For any integer  $d \geq 0$ ,  $s \geq 2$ , and any path  $P$ , we have:

$$g_T^{s,0}(P) \leq g_T^{s,d}(P) \leq (2d + 1)g_T^{s,0}(P)$$

*Proof.* The first part of the inequality following from the definitions, we only need to point out that a strategy with distance 0 can be obtained from a strategy at distance  $d$ , through the cost of replacing every guard by an interval of  $2d + 1$  guards following its movements.  $\square$

This observation leads us to focus on the asymptotic behaviour of  $g_T^s = g_T^{s,0}$  when  $T$  grow large, as it captures the behaviour of  $g_T^{s,d}$  for any fixed  $d, s$ . We also observe that within  $T$  steps, a spy walking on a path can reach at most  $2sT + 1$  different vertices, so that one can assume without loss of generality that the path has at most  $2sT + 1$  vertices and that the spy starts at its center or, similarly, that the path is an infinite path in both directions.

Precisely, we present upper and lower bounds for  $g_T^s(P)$  in case of any path  $P$ .

**Lemma 18.** *Let  $P$  be a path and let  $s, d$  be fixed integers. We have:*

$$g_T^s(P) \leq O(T^{1-1/(2s+1)})$$

*Proof.* To prove this result, we assume that  $P$  is infinite in both directions and we define from the spy's position at a current time  $t$  its *range* as the set of all  $2s(T - t) + 1$  vertices that it can reach by time  $T$ . We say that the spy's range is covered by a *meshing* of width  $w$  (or  $w$ -meshing) if any two consecutive guards are at distance  $w$  or  $w - 1$  from each other, in such a way that every vertex of the spy's range lies between the leftmost and the rightmost guards. It is said to be *regular* when all intervals have the same length.

We present a strategy that starts from a configuration in which the guards are arranged in a *regular meshing* of the spy's range of width  $w_0 = \lfloor T^{1/(2s+1)} \rfloor$ . Hence, the strategy will use  $O(2sT/w_0)$  guards, i.e.  $O(T^{1-1/(2s+1)})$  guards. At each turn, the strategy updates their positions in the following way:

- If the spy moved to the right (at any distance), we pick an interval of largest size and reduce it by moving right all guards standing on its left.
- Otherwise, we reduce this interval by moving left all guards standing on its right.

Following this strategy the distance between the leftmost and rightmost guards decreases by one at each turn, while the spy's range decreases by  $2s$ . Whenever all intervals are of equal length  $w$  and the guards cover more than  $2s(T - t) + 1 + w$  vertices, we can thus decide to fire (or disable) a guard that would not be needed to cover the spy's range, as it will never be needed again. We point out that with these rules the (remaining) guards form a meshing of the spy's range at any moment of the game, and that their number (and the size of the intervals) does *not* depend on the spy's behaviour, and is only a function of  $t$ . We denote by  $t_i$  the time at which the guards define a regular  $i$ -meshing (thus  $t_{w_0} = 0$ ), and so we can define constants  $\beta_i \leq 2$  such that the following equation holds:

$$\begin{aligned} t_w &= t_{w+1} + \overbrace{\left\lceil \frac{2s(T - t_{w+1})}{w} \right\rceil + 1}^{\text{number of intervals at } t_{w+1}} \\ &= t_{w+1} + \frac{2s(T - t_{w+1})}{w} + \beta_w \end{aligned}$$

Consequently,

$$\begin{aligned} (T - t_w) &= T - t_{w+1} - \frac{2s(T - t_{w+1})}{w} - \beta_{w+1} \\ &= (T - t_{w+1}) \left( 1 - \frac{2s}{w} \right) - \beta_{w+1} \end{aligned}$$

We now unroll the recurrence in order to obtain for any  $w$ :

$$(T - t_w) = T \cdot \prod_{w_0 \geq w' > w} \left( 1 - \frac{2s}{w'} \right) - \sum_{w_0 \geq w' > w} \beta_{w'} \cdot \prod_{w_0 \geq w'' > w'} \left( 1 - \frac{2s}{w''} \right)$$

We bring the reader's attention upon the fact that the last term is at most  $2w_0$  since  $\beta_{w'} \leq 2$  and the product is smaller than one. We now focus on the first term, which satisfies for every  $w$  such that  $2s < w < w_0 - 2s$ :

$$\prod_{w_0 \geq w' > w} \left(1 - \frac{2s}{w'}\right) = \prod_{w_0 \geq w' > w} \left(\frac{w' - 2s}{w'}\right) = \frac{w \times \cdots \times (w + 1 - 2s)}{w_0 \times \cdots \times (w_0 + 1 - 2s)}$$

Let us check that  $t_w \leq T$  for every  $w \geq 2 + 2s$  and thus that the strategy reaches a regular  $(2 + 2s)$ -meshing before turn  $T$  (bounding  $\beta_w$  by 2):

$$\begin{aligned} T - t_w &\geq T - t_{2+2s} \geq T \left(\frac{2}{w_0}\right)^{2s} - 2w_0 \geq T \left(\frac{2}{T^{1/(2s+1)}}\right)^{2s} - 2T^{1/(2s+1)} \\ &\geq T^{1/(2s+1)}(2^{2s} - 2) > 1 \end{aligned}$$

Consequently, the described strategy that uses  $O(T^{1-1/(2s+1)})$  guards, is able to cover the spy's range with a regular  $(2s + 2)$ -meshing after  $t_{2+2s} < T$  moves. By playing the very same strategy with each guard being replaced by an interval of  $4s + 3$  consecutive guards, one can instead be sure that every vertex of the spy's range hosts a guard at that time  $t_{2+2s}$ . The guards can then stay in place until time  $T$  is reached.  $\square$

**Lemma 19.** *For any  $T \in \mathbb{N}$  and any path  $P$  of length at least  $2sT$ , and for any  $s \geq 2$ ,*

$$g_T^s(P) \geq T^{\lceil \log_{\frac{2s}{s-1}} T \rceil}.$$

*Proof.* In  $T$  turns, the spy can reach a vertex at distance at most  $sT$  from  $v_0$ . Therefore, we can restrict our study to a finite path of length  $2sT$  with central vertex  $v_0$ . Precisely, let  $P = (\ell = v_{-sT}, \dots, v_{-1}, v_0, v_1, \dots, v_{sT} = r)$ . W.l.o.g., let us assume that, after the guards have chosen their initial positions, there are no more guards at the right to  $v_0$  (i.e., in  $(v_1, \dots, r)$ ) than at its left.

Let  $X$  be the smallest integer strictly greater than  $\frac{s+1}{2s}T$ .

Consider the strategy of the spy where it goes at full speed to the “right” during the first  $X$  turns. After this step, the spy occupies  $v_{sX}$  and it remains  $T - X$  turns. That is, during the remaining steps, the only vertices that could be reached by the spy belong to  $P_r = \{v_{s(2X-T)}, \dots, v_{sT}\}$ . This implies that, any guard that could reach the spy during the last  $T - X$  turns must have initially occupied a vertex in  $(v_{s(2X-T)-T}, \dots, r)$ . Since, by definition of  $X$ ,  $s(2X - T) - T > 0$ , no guards initially occupying a vertex in  $(\ell, \dots, v_0)$  can be used to win against this strategy of the spy. Therefore, only the guards initially occupying vertices in  $(v_1, \dots, r)$  can be used during the remaining  $T - X$  turns. In particular, it implies that there must be at least  $g_{T-X}^s(P) \geq g_{\lfloor \frac{s-1}{2s}T \rfloor}^s(P)$  such guards.

Since these guards are distinct from the ones that were occupying  $(\ell, \dots, v_0)$  initially, and since there were no more guards on the right side, it follows that  $g_T^s(P) \geq 2 \cdot g_{\lfloor \frac{s-1}{2s}T \rfloor}^s(P)$ . Note also that  $g_1^s(P) \geq 1$ . Therefore, by induction on  $T$ :

$$g_T^s(P) \geq 2^{\lceil \log_{\frac{2s}{s-1}} T \rceil} = T^{\lceil \log_{\frac{2s}{s-1}} T \rceil}$$

$\square$

## 5 Conclusion

Many open questions remain such as the characterization of the guard-number in other graph classes, e.g., in grids. The exact complexity of the problem of deciding it is also still open. Finally, the latter variant with a countdown seems much more difficult to study and almost nothing is known about it.

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